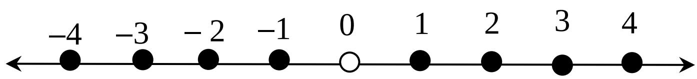
2. INEQUALITIES AND ABSOLUTE VALUES

§2.1. The Ordering of the Real Numbers

In addition to the arithmetic structure of the real numbers there's the order structure. The real numbers can be represented by points on a line and if one point is to the left of another we say that the corresponding number is smaller, or less, than the other.



Alternatively we can call the numbers on the right of zero the **positive** numbers and then define ***x* < *y*** if $x = y + z$ for some positive z. Numbers to the left of zero are called **negative**.

We define ***x* ≤ *y*** to mean that $x < y$ or $x = y$. Of course if x and y were specific numbers we'd know which is the case and would write $x < y$ or $x = y$ instead of the more uncertain $x \le y$. But often we don't know which is the case.

Sometimes it's convenient to specify the larger number first and write $\boldsymbol{x \geq y}$ instead of $y \leq x$. Finally, $\boldsymbol{x > y}$ means the same as $y < x$.

Example 1: $2 \leq 4$. In fact we can be more specific by writing $2 < 4$. These can be written alternatively as $4 \geq 2$ and $4 > 2$ respectively.

Inequalities look rather like equations and in many ways we can operate with them the way we would equations. But beware. We'll later see that there are some important differences.

The relation $\leq$ has three important properties that are obvious when you consider the number line.

Reflexive Property: $x \leq x$ for all x.
Anti-Symmetric Property: If $x \leq y$ and $y \leq x$ then $x = y$.
Transitive Property: If $x \leq y$ and $y \leq z$ then $x \leq z$.

The first of these is so obvious it's barely worth mentioning. The second is occasionally useful as a way of proving that two numbers are equal. The third is something we use all the time.

There are other orderings in mathematics where these three properties hold. For example, if S and T are sets the

statement S is a **subset** of T means that every element of S is an element of T. We write $S \subseteq T$. For example the set of boys in a family is a subset of the set of children in the family.
The analogy is made clear by the fact that the symbol we use for subset is very similar to the one we use for 'less-than-or-equals'. Moreover the reflexive, anti-symmetric and transitive properties hold for subsets.
Reflexive Property: $S \subseteq S$ for all S.
Anti-Symmetric Property: If $S \subseteq T$ and $T \subseteq S$ then $S = T$.
Transitive Property: If $S \subseteq T$ and $T \subseteq W$ then $S \subseteq W$.

Another important property of the ordering of real numbers is the following.

Total Order Property: For any two distinct real numbers either $x < y$ or $y < x$.

The real numbers are arranged linearly and any two such numbers can be compared. Notice that the subset ordering doesn't have this property. It is possible to have two distinct sets S, T where neither is a subset of the other. For example, the set of positive numbers is not a subset of the set of negative numbers, nor vice versa.

§2.2. Inequalities

An algebraic inequality is a statement in which there are two algebraic expressions separated by one of the following: ≤, <, ≥, >.

Example 2: Examples of inequalities are the following:
(1) $2x + 5 < x^2$;
(2) $x^2y \geq x + 2y$.

Inequalities look very much like equations and we can work with them like equations in many ways. The next theorem uses the fact that the sum of two positive numbers is positive.

Theorem 1:
If $a \leq b$ and $c \leq d$ then $a + c \leq b + d$.
Proof: Suppose $a \leq b$ and $c \leq$ d. Then $b - a \geq 0$ and $d - c \geq 0$.
Then $(b + d) - (a + c) = (b - a) + (d - c) \geq 0$. ✋☺

Note that this doesn't work for subtraction. That is, just because $a \leq b$ and $c \leq d$ it doesn't follow that $a - c \leq b - d$. [For example $12 \leq 15$ and $3 \leq 10$ but $12 - 3 \leq 15 - 10$ is FALSE.

Similar results hold for the three other types of inequality. When it comes to multiplying or dividing inequalities we must be careful. We can multiply or divide both sides of an inequality by a positive number.

Theorem 2: If $a \le b$ and $x > 0$ then $ax \le bx$.
Proof: Suppose that $a \le b$ and x > 0. Then $b - a \ge 0$ and so $bx - ax = (b - a)x \ge 0$.
Thus $ax \le bx$. ✋☺

The same is true for dividing by a positive number because if x > 0 then 1/x > 0 and dividing by x is the same as multiplying by $1/x$. Just remember that we must multiply or divide both sides by a positive number.

Example 3: If 3 < 5 is $3x < 5x$?
Answer: If x is positive this is so, but suppose $x = -2$.
Is –6 < –15? No! It is the other way around: –15 < –6.

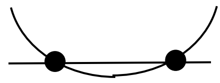
When you multiply by a negative number the inequality reverses direction. And if you don't know whether x is positive or negative you're stuck. You can't do anything without considering cases.

One of the standard things to do is to solve an inequality, that is to find the range of values of the variables for which the inequality holds. Here we consider only inequalities that inolve just one variable.

Example 4: Solve the inequality $3x + 5 < 7x - 19$.
Solution: Subtract $3x + 5$ from both sides to get $0 < 4x - 24$, that is $4x > 24$.
We now divide both sides by 4 to get $x > 6$.
So the solution is $\{x \mid x > 6\}$.

Example 5: Solve the inequality $x^2 + 6 \leq 5x$.
Solution: Subtract $5x$ from both sides to get $x^2 - 5x + 6 \leq 0$.
Now factorise $x^2 - 5x + 6$ to get $(x - 2)(\underline{x} - 3) \leq 0$.
This is exactly what we might have done if we were solving the equation $x^2 + 6 = 5x$.
If $(x - 2)(x - 3) = 0$ then indeed we get $x = 2$ or 3, but what if $(x - 2)(x - 3) < 0$?
Clearly this can only happen if the factors have opposite signs.
So either $x - 2 > 0$ and $x - 3 < 0$, in which case $2 < x < 3$,
or $x - 2 < 0$ and $x - 3 > 0$ in which case $x < 2$ and $x > 3$.
The latter case is impossible. A number can't be less than 2 and at the same time be greater than 3. So we're left with just $2 < x < 3$. Including the endpoints where we get equality and the complete solution is $\{x \mid 2 \leq x \leq 3\}$.

If we sketch the parabola $x^2 - 5x + 6 = (x - 2)(x - 3)$ we can see clearly what is going on.



§2.3. Intervals

In each of the above cases the answer has been an interval. An **interval** on the real line is a subset S such that if $x, y \in S$ with $x < y$ then S contains every real number between them. The whole real line is clearly an interval, as is a set $\{a\}$ containing just one number. (The definition says that if there are two different numbers in the set the set has to contain every number between them. It doesn't say that there have to be two distinct numbers in the set.) These are the extreme types of interval. All intervals have to be one of the following ten types:

$\mathbb{R}$

$\{a\}$

$[a, b] = \{x \mid a \le x \le b\}$;

$[a, b) = \{x \mid a \le x < b\}$;

$(a, b] = \{x \mid a < x \le b\}$;

$(a, b) = \{x \mid a < x < b\}$;

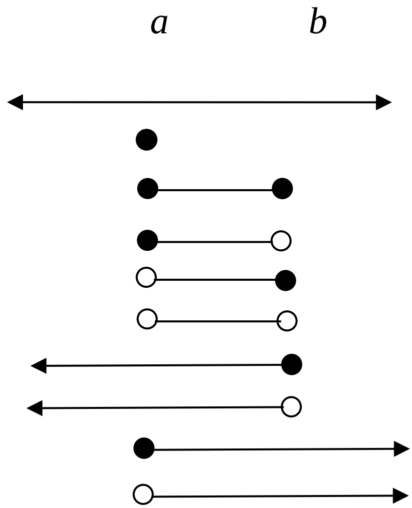
$(-\infty, a] = \{x \mid x \le a\}$;

$(-\infty, a) = \{x \mid x < a\}$;

$[a, \infty) = \{x \mid x \ge a\}$;

$(a, \infty) = \{x \mid x > a\}$;

We can represent these by means of diagrams:



$\mathbb{R}$
$\{a\}$
$[a, b]$
$[a, b)$
$(a, b]$
(a, b)
$(-\infty, a]$
$(-\infty, a)$
$[a, \infty)$
(a, ∞)

The answer to Example 4 can be written as $(6, \infty)$ and the answer to Example 5 as $[2, 3]$.

§2.4. Harder Inequalities

Example 6: Solve the inequality $\dfrac{x+1}{2x+6} > \dfrac{x+2}{3x+7}$.

Solution: If this was the equation $\dfrac{x+1}{2x+6} = \dfrac{x+2}{3x+7}$ we'd simple 'cross multiply', that is multiply both sides by the product of the denominators, to get

$(x + 1)(3x + 7) = (x + 2)(2x + 6)$

$\therefore\ 3x^2 + 10x + 7 = 2x^2 + 10x + 12$

$\therefore\ x^2 = 5$, giving $x = \pm\sqrt{5}$. Can we do something similar here?

The answer is no, at least not without a lot of tedious case-splitting. We are permitted to 'cross multiply' but if the product of the denominators is negative the inequality changes from < to >. Let's see how this case-splitting would work.

Case I: $(2x + 6)(3x + 7) > 0$: That will come about if both factors are positive or both are negative.

Case IA: $2x + 6 > 0$ and $3x + 7 > 0$:

Here $x > -3$ and $x > -\frac{7}{3}$. Since $-3 < -\frac{7}{3}$ both inequalities are satisfied when $x > -\frac{7}{3}$.

In this case we can cross-multiply to get:

$$(x + 1)(3x + 7) > (x + 2)(2x + 6)$$

$\therefore\ 3x^2 + 10x + 7 > 2x^2 + 10x + 12$

$\therefore\ x^2 > 5$, giving $x < -\sqrt{5}$ or $x > \sqrt{5}$.

We have to combine these with $x > -\frac{7}{3}$.

Now $\frac{7}{3}$ is about 2.33 and $\sqrt{5}$ is about 2.24 so plotting $-\sqrt{5}$, $\sqrt{5}$ and $\frac{7}{3}$ on the number line we get:

so if $x > -\frac{7}{3}$ and $x < -\sqrt{5}$ we get the interval $\left(-\frac{7}{3}, -\sqrt{5}\right)$
and if $x > -\frac{7}{3}$ and $x > \sqrt{5}$ we get the interval $(\sqrt{5}, \infty)$.
The solution in this case is the union of these intervals which we write as $\left(-\frac{7}{3}, -\sqrt{5}\right) \cup (\sqrt{5}, \infty)$.

Case IB: $2x + 6 < 0$ and $3x + 7 < 0$:
Here $x < -3$ and $x < -\frac{7}{3}$. Since $-3 < -\frac{7}{3}$ both inequalities are satisfied when $x < -3$.
In this case we can cross-multiply, again getting $x^2 > 5$, giving $x < -\sqrt{5}$ or $x > \sqrt{5}$.
We have to combine these with $x < -3$.
Now $x < -3$ is incompatible with $x > \sqrt{5}$ and if $x < -\sqrt{5}$ and $x < -3$ we simply have $x < -3$.
So the solution in this case is $x < -3$.

Case II: $(2x + 6)(3x + 7) < 0$: That will come about if one factors is positive and the other is negative.
Case IIA: $2x + 6 > 0$ and $3x + 7 < 0$:
Here $x > -3$ and $x < -\frac{7}{3}$. So in this case $-3 < x < -\frac{7}{3}$.
In this case we can cross-multiply but we get:

$$(x + 1)(3x + 7) < (x + 2)(2x + 6)$$

$\therefore\ 3x^2 + 10x + 7 < 2x^2 + 10x + 12$
$\therefore\ x^2 < 5$, giving $x \in (-\sqrt{5}, \sqrt{5})$.

We have to combine this with $x \in (-3, -\frac{7}{3})$. But this is disjoint with $(-\sqrt{5}, \sqrt{5})$. That is there are no numbers in common. So case IIA cannot arise.

Case IIB: $2x + 6 < 0$ and $3x + 7 > 0$:

Here $x < -3$ and $x > -\frac{7}{3}$. Again this case cannot arise.

Putting all these cases together we get the solution

$$\left(-\frac{7}{3}, -\sqrt{5}\right) \cup (\sqrt{5}, \infty) \cup (-\infty, -3).$$

Are you still with me? This method involves endless case-splitting and then having to combine the resulting inequalities, keeping track of which ones are 'ands' and which are 'ors'. There has to be a better way! And there is.

Example 6 (again): Solve the inequality $\frac{x+1}{2x+6} > \frac{x+2}{3x+7}$

.

Solution: We subtract to get $\frac{x+1}{2x+6} - \frac{x+2}{3x+7} > 0$.

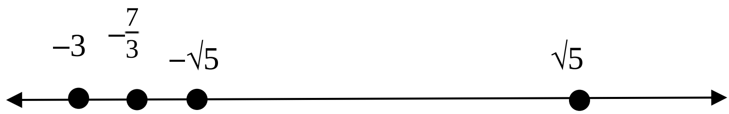
Putting over a common denominator we get

$$\frac{(3x^2 + 10x + 7) - (2x^2 + 10x + 12)}{(2x + 6)(3x + 7)} > 0.$$

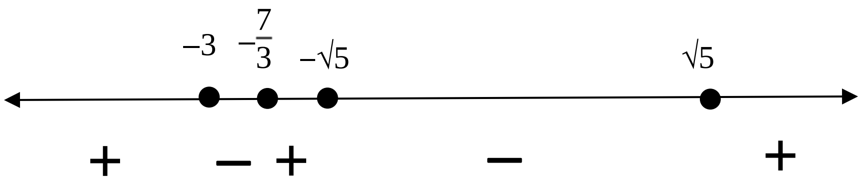
$$\therefore \frac{x^2 - 5}{(2x+6)(3x+7)} > 0.$$

$$\therefore \frac{(x - \sqrt{5})(x + \sqrt{5})}{(2x+6)(3x+7)} > 0.$$

Plotting the four number $\pm\sqrt{5}$, -3 and $-\frac{7}{3}$ on the number line we get



If $x > \sqrt{5}$ all four factors are positive and the inequality holds. As x moves left, every time it passes one of these four marked points one of the factors becomes negative. The sign of $\frac{(x - \sqrt{5})(x + \sqrt{5})}{(2x+6)(3x+7)}$ alternates between positive and negative as indicated below.



So the solution set is $(-\infty, -3) \cup \left(-\frac{7}{3}, -\sqrt{5}\right) \cup (\sqrt{5}, \infty)$ as before.

Theorem 3: If $0 \le a < b$ then $a^2 < b^2$.
Proof: Suppose $0 \le a < b$.
Then $b^2 - a^2 = (b - a)(b + a) > 0$ since both factors are positive. Hence $a^2 < b^2$. ✋☺

Example 7: Solve the inequality $\sqrt{5x - 6} < x$.

Solution: To begin with we must have $x \ge \frac{6}{5}$ for the square root to exist. Also, since √ denotes the non-negative square root we must have $x > 0$. If $x \ge \frac{6}{5}$ then the right hand side is positive so we can square both sides.

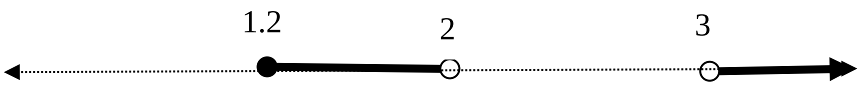
$\therefore\ 5x - 6 < x^2$ and so $x^2 - 5x + 6 > 0$, that is:

$$(x - 2)(x - 3) > 0.$$

This is satisfied when $x < 2$ and also when $x > 3$.

But since $x \ge \frac{6}{5}$ this reduces to

$$[6/5, 2) \cup (3, \infty).$$



Example 8 (Hard): Solve $\sqrt{x - 2} + \sqrt{x + 5} < 3$.
Solution: For the square roots to exist we must have $x \ge 2$. Since both sides will be positive we can square both sides, getting:

$$(x - 2) + (x + 5) + 2\sqrt{(x-2)(x+5)} < 9.$$

$$\therefore \sqrt{(x-2)(x+5)} < \frac{6 - 2x}{2} = 3 - x.$$

We can conclude from this that $x \le 3$ but we can make the interval narrower.

We are entitled here to square both sides again, getting:

$$(x - 2)(x + 5) < x^2 - 6x + 9.$$

$\therefore\ x^2 + 3x - 10 < x^2 - 6x + 9.$

$\therefore\ 9x < 19.$

$\therefore\ x < \frac{19}{9} = 2\frac{1}{9}.$

The solution set is $[2, 2\frac{1}{9})$.

§2.5. Absolute Values

The **absolute value** of a real number x is the magnitude of x, ignoring the sign. It is denoted by $|x|$. If x is positive then $|x| = x$ and if x is negative then $|x| = -x$. And, of course $|0| = 0$.

We can define the absolute value of x very compactly as $\sqrt{x^2}$, remembering that √ denotes the positive square root. But, of course, it would be silly to find the absolute value this way!

Clearly $|xy| = |x|.|y|$ for all real numbers x, y but things don't work out quite so neatly for sums. It's NOT true in general that $|x + y| = |x| + |y|$. If x, y have opposite signs then $|x + y|$ will be less than $|x| + |y|$. For example. $|3 + (-1)| = 2$ while $|3| + |-1| = 4$. All we can say for sums is that $|x + y| \le |x| + |y|$.

Theorem 4 (Triangle Inequality): $|x + y| \le |x| + |y|$ for all real numbers x, y.

Proof: We could prove this by examining various cases but the simplest proof is the following.

First note that $xy = \pm|x|.|y| \le |x|.|y|$.

$$\begin{aligned} \text{Hence } (x + y)^2 &= x^2 + y^2 + 2xy \\ &= |x|^2 + |y|^2 + 2xy \\ &\le |x|^2 + |y|^2 + 2|x|.|y| \\ &\le (|x| + |y|)^2. \end{aligned}$$

Taking positive square roots we get $|x + y| \le |x| + |y|$. ✋☺

We get equality unless x, y have opposite signs. The reason for calling this result the **Triangle Inequality** is that the corresponding result for complex numbers (where numbers live in a plane and $|z|$ denotes the length of the line joining 0 to z) can be interpreted as saying that the length of any side of a triangle is less than or equal to the sum of the lengths of the other two sides.

Example 9: Solve the equation $|3 - 2x| = |x + 1|$.
Solution: Here case splitting is not such a bad option.
Case I: $x \leq -1$: The equation becomes $3 - 2x = -x - 1$, so $x = 4$. This lies outside the range for this case so we reject it.
Case II: $-1 < x < \frac{3}{2}$: The equation is now $3 - 2x = x + 1$, so $3x = 2$ and so $x = \frac{2}{3}$.
Case III: $x \geq \frac{3}{2}$: We solve $2x - 3 = x + 1$, getting $x = 4$, which this time we accept.
So the solutions are $x = \frac{2}{3}$ and $x = 4$.

However a better option is to square both sides, removing the absolute value signs.
$(3 - 2x)^2 = (x + 1)^2$, giving $4x^2 - 12x + 9 = x^2 + 2x + 1$ or:

$$3x^2 - 14x + 8 = 0.$$

$\therefore\ (3x - 2)(x - 4) = 0$ and so $x = \frac{2}{3}$ or 4.

Example 10: Solve the inequality $|3 - 2x| > |x + 1|$.
Solution: Since both sides are positive we may square:

$$(3 - 2x)^2 > (x + 1)^2.$$

$\therefore\ 4x^2 - 12x + 9 > x^2 + 2x + 1.$
$\therefore\ 3x^2 - 14x + 8 > 0.$
$\therefore\ (3x - 2)(x - 4) > 0.$

$\therefore\ x > \frac{2}{3}$ and $x > 4$, in which case $x > 4$,

or $x < \frac{2}{3}$ and $x < 4$, in which case, $x < \frac{2}{3}$.

So the solution is $(-\infty, 2/3) \cup (4, \infty)$.

Example 11: Solve the inequality $|3 - x| > 2 - |x + 1|$.

Solution: We can't square both sides because the right hand side could be negative.

But we can rewrite the equation as $|x - 3| + |x + 1| > 2$ and now it's permissible to square both sides getting:

$$(x - 3)^2 + (x + 1)^2 + 2|(x - 3)(x + 1)| > 4.$$

$\therefore\ 2x^2 - 4x + 10 + 2|(x - 3)(x + 1)| > 4.$

$\therefore\ x^2 - 2x + 3 + |(x - 3)(x + 1)| > 0.$

We could square both sides again but that would give us an inequality involving x^4. Not pretty! Instead we fall back on good old case splitting.

Case I: $(x - 3)(x + 1) \geq 0$:

Clearly we have $x \leq -1$ or $x \geq 3$ in this case.

The equation reduces to $x^2 - 2x + 3 + x^2 - 2x - 3 > 0$.

$\therefore\ 2x^2 - 4x > 0.$

$\therefore\ x(x - 2) > 0$, which gives $x < 0$ or $x > 2$.

We have to intersect this region with the region for case I. That is, $x \leq -1$ or $x \geq 3$ and $x < 0$ or $x > 2$.

The former region lies within the latter so the solution in this case is $x \leq -1$ or $x \geq 3$.

In other words it is $(-\infty, -1] \cup [3, \infty)$.

Case II: $(x - 3)(x + 1) < 0$: This is the case $-1 < x < 3$.
The equation reduces to $x^2 - 2x + 3 - x^2 + 2x + 3 > 0$.
$\therefore\ 6 > 0$.
But this is always true so the entire region for case II is in the solution set. This is the interval $(-1, 3)$.

Combining the solutions for the two cases we get $(-\infty, -1] \cup (-1, 3) \cup [3, \infty)$. But this is the entire real line! Do you get the feeling that there has to be a simpler method? Let's try case-splitting from the very beginning.

Write the inequality as $|x - 3| + |x + 1| > 2$.
Case I: $x \le -1$: The equation becomes $3 - x - x - 1 > 2$, that is, $x < 0$. So the entire case I is in the solution set.
Case II: $-1 < x < 3$: The equation becomes:
$3 - x + x + 1 > 2$, that is, $2 > 0$. So the entire case II is in the solution set.
Case III: $x \ge 3$: The equation becomes $x - 3 + x + 1 > 2$, that is, $x > 0$. So the entire case III is in the solution set.
Clearly the solution set is the whole real line.

Hmm! There might be an even simpler method.
Consider $|(x - 3) - (x + 1)|$.
On the one hand this is $|-4| = 4$.
But the Triangle Inequality gives:
$|(x - 3) - (x + 1)| \le |x - 3| + |x + 1|$.
So in fact $|x - 3| + |x + 1| \ge 4$ for all x.